Vertor Fields in Space Multivariable Calculus X. Du

- Useful in describing:
  - Flow of a fluid
  - Force fields (electromagnetic fields, gravitational fields, etc.)
- In two variables: $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$
- In three variables: $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$
- **Gradient** vector field: If $f(x_1, x_2, \ldots x_n)$, then $\nabla f = \langle f_{x_1}, f_{x_2}, \ldots f_{x_n} \rangle$ is the gradient vector field of $f(x_1, x_2, \ldots x_n)$.
  - For a function of three variables $f(x,y,z)$, $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$
- **Conservative vector field**: A vector field $\vec{F}$ is **conservative** if there exists a **potential function** *f* such that $\nabla f = \vec{F}$.
  - Most prominent example: gravity
  - $\nabla \times \vec{F} = \vec{0}$ (assuming continuous second order partials of *f*)
- **Divergence**: $\nabla \cdot \vec{F}$
  - For a vector field of three variables $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$, $\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$
  - Measures the amount of stuff flowing out of a region
- **Curl**: $\nabla \times \vec{F}$
  - Only for a vector field in three-space: $\vec{F}(x,y,z) = \langle P, Q, R \rangle$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

  - Measures the amount of rotation in a vector field.
  - Use right hand rule to determine direction of curl.
  - If $f(x,y,z)$ has continuous second order partials, then $\nabla \times (\nabla f) = \vec{0}$.
  - $\nabla \cdot (\nabla \times \vec{F}) = 0$ (assuming continuous second order partials)
- **Laplacian**: $\nabla^2 f = \nabla \cdot \nabla f$
  - For a function of three variables $f(x,y,z)$, $\nabla^2 f = \frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y} + \frac{\partial^2 f}{\partial^2 z}$
  - Measures the “curvature” of a multivariable function, or how much a point is different in its neighborhood (i.e. how “sharp” a point is on the function).